A Cube Tiling with Non-periodic Planes

J.M. Taylor

23rd September, 2011

Introduction

We present a tiling of Euclidean three-space by cubes whose faces are decorated with square tiles known to produce only non-periodic tilings of the plane. The square tiles used are equivalent to C. Goodman-Strauss' Trilobite and Cross tiles though we call them Arrow and Cross tiles. We do not prove details of the possible tilings by these square tiles here but note that one feature of the 2D tilings, the possible fault-line, is not supported on the 3D application. A detailed 3D tiling demonstration follows using an "offset" method referred to central tiles of certain infinite structures. The structures are stacked in the $(1, 1, 1)$ and $(-1, -1, -1)$ directions from the centre of the 3D tiling, permitting a translation of the cube tiling along one direction. We note that no worse translation is possible while conforming to the matching within planes. Lastly a shape alone version of the cube is described.

Five image files accompany this text.

Cube1. The limit-periodic 2D tiling,
Cube2. The 2D and 3D tiles and a fault-line,
Cube3. No fault-line in 3D tiling,
Cube4. The offset method and global structure and
Cube5. Shape alone matching of the cube.

A Constraint on Tilings of the Plane

A tiling by the Arrow and Cross tiles is depicted in the image file Cube1, while the tiles and their application to the cube are found in Cube2. The ratio of Arrow to Cross tiles in a 2D tiling is two as to one and the cube has two Cross faces and four Arrow faces. The matching of the faces is carried out exactly as in the plane, i.e. coloured stripes must match across vertices and an 'out' arrow must match an 'in' arrow. It can be observed that the cube has identical faces on opposite sides with corresponding markings directly opposite each other. Hence the underside of a slab of correctly matched cubes will support a mirror-image tiling compared to the top of the slab.

It is not the case, however, that every legal 2D tiling can be found on a plane of correctly matched cubes in a 3D tiling. In two dimensions the matching of the square tiles admits a single infinite fault-line similar to the fault-lines of R.

Robinson's six square tile set. The fault-line cannot occur in a 3D cube tiling because of matching constraints operating on orthogonal planes.

The smallest scale of fault-line is depicted in Cube2. The fault-line runs along an infinite string of Arrow tiles following head to tail. In the first row on either side there is a misalignment of Cross tiles with Arrow tiles, compared with the regular, limit-periodic structure in Cube1. A Cross tile is opposite an Arrow pointing towards the fault-line, whereas the regular tiling has Cross tile opposite Cross tile or a pair of Arrow tiles pointing towards each other.

The fault-line divides the tiling into two half-planes, each otherwise a regular limit-periodic tiling. The nearest misalignment can occur on any scale of the lattice structure, i.e. at any distance 2^k, k a non-negative integer, from a fault-line. Goodman-Strauss describes a method of realignment of misaligned half-planes by a series of shifts of one half-plane along the fault-line. At the first stage of the process he corrects the nearest misalignment, then the next-nearest remaining misalignment, etc.

We show in Cube3 that no misalignment can occur at distance $2^0, 2^1, 2^2, 2^3$ or any 2^k distance from the fault-line, being constrained by matching in an orthogonal plane. We proceed by trying to form a misaligned pair, Cross and Arrow, either side of a proposed fault-line. We do this on the top faces of the bottom row of a stack of cubes in a roughly triangular shape. The top face of the central cube in the bottom row is on the proposed fault-line and we try to make the faces marked x, at 2^k distance on either side, misalign. The front faces below a row force the front faces in that row. The front faces of the cubes stacked on top of the proposed fault-line are constrained to point upwards until the next sized concentric ring is reached. So the faces marked x are necessarily co-ordinated by stripe matching to be either two Arrows pointing towards the would-be fault-line or two Crosses. See Cube3 for the construction.

3D Tiling Demonstration

For the purposes of this presentation the tilings of the planes of the cubes may be considered to come in two forms: a tiling with a Cross at the centre inside ever larger blue rings or one with an Arrow at the centre inside ever larger blue rings. The first tiling is depicted in the image file Cube1.

It is easy to modify the Cross-centred tiling in ones imagination to become the Arrow-centred tiling. After swapping the central tile (heavily outlined), the stripe pairs that depart the central tile and are transmitted diagonally through infinite sequences of Cross tiles must be modified to agree with the central Arrow tile. Similarly, the infinite sequences of Arrow tiles , head to tail, that run vertically and horizontally along the axes must also be made to agree with the central Arrow tile.

The assumed 3D tiling structure is indicated on the right in the image file Cube4. The central tiles of the 2D tilings are found on the faces of the cubes central to six 'arms' of infinite length. These central cubes are at positions (k, k, k). The $(1, 1, 1)$ translation applies to the 3D tiling as a whole and hence to the six-armed structures that form its framework. The Arrow tiles along the horizontal and vertical axes of a 2D tiling decorate the exterior of the arms. The

Cross tile decorates the upper and lower surfaces of the cubes that constitute the vertical arms, and the Cross is also hidden from view inside the four horizontal arms. The Crosses inside the arms lie on the diagonals of 2D tilings.

To describe an arbitrary cube tile in the 3D tiling we reference it from the central cube, which we designate to be at $(0, 0, 0)$, and is depicted on the left in the image file Cube4. Note that the xy plane is rotated from the usual to facilitate the description. The 'right' face lies in a plane parallel to the yz plane, the 'front' face parallel to the xz plane and the 'top' face parallel to the xy plane. The tilings of the plane made with right faces have an Arrow tile pointing down as the central tile; tilings of the plane made with front faces have an Arrow pointing up as the central tile and tilings of the plane made with top faces have a Cross as the central tile. Recall that the cube has the same markings on opposite faces so that describing the faces about one corner effectively describes the whole cube.

To find the markings on the cube at $(x, y, z) = (3, 5, 7)$ we consider the offsets from the central tile of the plane for each face. For example the central tile of the plane containing the right face is on the cube at $(3, 3, 3)$ and we need the offsets $y - x = 5 - 3 = 2$ in the horizontal and $z - x = 7 - 3 = 4$ in the vertical, i.e. co-ordinates $(2, 4)$ on the 2D tiling with central tile at $(0, 0)$. We simply consult the appropriate tiling of the plane for a right face (central tile is Arrow pointing down) and count from the central tile 2 tiles to the right and 4 tiles up to find the decoration for this right face. The tables in Cube4 gives the 2D co-ordinates for the faces: right, $(y-x, z-x) = (2, 4)$, offset from $(3, 3, 3)$; front, $(x - y, z - y) = (-2, 2)$, offset from $(5, 5, 5)$, and top, $(x - z, y - z) = (-4, -2)$, offset from $(7, 7, 7)$. For the front face we consult the tiling of the plane with an Arrow pointing up in the central position and for the top face the tiling with a Cross in the central position. The orientation of a central Cross tile is arbitrary but we choose them all alike.

Now we need to check that the three face decorations found always comprise a valid corner of a cube. First it is easily checked that, apart from a central tile, the Cross tile is always and only found at positions $(\pm 2^r(2p + 1), \pm 2^r(2q + 1))$, p, q, r non-negative integers . We show by contradiction that the three offsets cannot comprise a set with none having this property, i.e. none with both co-ordinates having the same power of 2 component. Then we find that when one offset has the property the other two do not, in fact that they are opposite Arrows exactly as required. We know that the cubes belonging to the arms of the 3D tiling have the required decorations and conform to the offset method. These cubes and only these have two of x, y and z the same and the third different and therefore we may ignore offsets with a zero co-ordinate and those with two co-ordinates the same in the following proofs.

We assume, then, for any set of three offsets designating a cube corner (not on an arm) that no pair of these 2D co-ordinates have the same power of 2 component, i.e. we assume no offset indicates a Cross decoration. It is necessary to use three different formulas, up to sign, to describe the offset components, to achieve this.

Let

$$\begin{aligned}
(y-x, z-x) &= (2^i f, 2^j g),\\
(x-y, z-y) &= (-2^i f, 2^k h),\\
(x-z, y-z) &= (-2^j g, -2^k h),
\end{aligned}$$

i, j, k, non-negative integers, all different; f, g, h, odd integers.

Then $(y-x)+(z-y) = z-x = 2^i f + 2^k h = 2^j g$.

Allowing $i > k$ we have $2^k(2^{i-k} f + h) = 2^j g$ and we find that $j = k$ (and by the symmetric argument when $k > i$, $j = i$), contradicting the assumption that i, j and k are all different. Therefore any cube corner found by the offset method has at least one Cross tile on a component face.

Next we show that the existence of one Cross on a corner implies two opposite Arrows pointing towards and away from the Cross on the other two faces. With a Cross at offset $(a, b) = (\pm 2^r(2p+1), \pm 2^r(2q+1))$, p, q, and r non-negative integers and $a \neq b$ (We continue to ignore the cubes on the arms.) we have the three cases tabulated below, e.g. When the Cross tile is on the right face, its offset $(y-x, z-x) = (a, b)$, the front face has offset $(x-y, z-y) = (-a, -(a-b))$ and the top face has offset $(x-z, y-z) = (-b, a-b)$.

	right	*front*	*top*
1.	(a, b)	$(-a, -(a-b))$	$(-b, a-b)$,
2.	$(-a, -(a-b))$	(a, b)	$(a-b, -b)$,
3.	$(-(a-b), -a)$	$(a-b, -b)$	(a, b).

Taking the first case, right face with a Cross tile, first we note that $-a$ and $-b$ have at least one less power of 2 as a factor than has $a-b$ (making $a-b$ necessarily even) and therefore both front and top are Arrow tiles (seeing they are not Cross tiles). Next we note the mirror symmetry of Arrow tiles in the horizontal axis (and in the vertical axis) in a 2D tiling. Thus the rows numbered $-(a-b)$ of the front and $(a-b)$ of the top have the same pattern of Arrows along them. These Arrows point parallel to the rows precisely when the position value (first co-ordinate value for the plane) has a power of 2 component less than that of the row value (second co-ordinate value). Then it can be observed that these Arrows along the rows change direction with the spacing $|a-b|$, the difference between $-a$ and $-b$. (The value of $|a-b|$ is expressible in odd multiples of a power of 2. The Arrows change direction on the spacing of the power of 2 and therefore also on an odd multiple of it.) So we have two Arrows, one pointing left and one right, as required to decorate the front and top faces of the cube with a Cross on its right face.

When the Cross is on the top face, case 3., $-(a-b)$ and $a-b$ are column numbers and, by similar arguments, the resulting Arrow tiles point up and down. The second case, a Cross on the front face, combines a row and a column number that unite a left with an up Arrow tile or a right with a down Arrow tile, as required.

It remains to be observed that there are four possible ways to orient the cube with respect to coloured stripes on the Cross tile and each can be made to agree with the given Arrow tiles on adjacent faces of the cube. The cube can be rotated through π about an axis through the centre of the two Cross faces and the Arrow tiles will coincide with the previous position but the stripes will belong to the opposite corner. The alternate pair of corners can be presented by turning the cube over and using the other Cross. Thus whatever orientation of the cube matches the tiling of the plane is achievable without disturbing the Arrow tiles on the other two faces of the cube corner. This completes the proof that the assumed 3D structure is possible with the cube tile.

Lastly we show that no worse translation of the space than a translation in the $(1, 1, 1)$ direction is possible. Translation in one orthogonal direction, say the $(0, 1, 0)$ direction, makes the yz plane and the xy plane consist of 2D tilings that are periodic in one direction, which is impossible. And translation in two orthogonal directions, say the $(0, 1, 1)$ direction, leaves the yz plane consisting of strips repeated with a translation along their infinite length, again not a possible 2D tiling. Therefore we need the translation to be in three orthogonal directions to preserve the non-periodicity of the tilings on the planes.

Shape-alone Matching

The cube tiling demonstration is completed by exhibiting a method of matching the cubes by shape alone. We recall that both the arrow markings and the coloured stripes are the same on opposite faces of the cube, e.g. a red stripe on the top face is directly above a red stripe on the bottom face and an arrow head on the top face is directly above an arrow head on the bottom face. This allows us to place the matching shapes corresponding to marks on top and bottom faces between these identical marks on the sides of the cube.

For this discussion we assume a vantage point on top of the cube at a corner looking down. (See the image file Cube5.) Each such corner has below it, passing from lower left to upper right, a hole and it has projecting from upper left and from lower right two rods angled to fit into similar holes in adjacent cubes. These rods transmit the coloured stripe information across vertices. The rod comes from lower right for the stripe on the right and from upper left for the stripe on the left. To distinguish red from blue we use a rod shape with a horizontal join for blue and a rod shape with a vertical join for red. Thus red and blue will not fit together through the hole in the intervening cube. These arrangements are the same at every edge of the cube.

The arrow markings are transmitted by a pair of bumps or dents on the face between the identical marks. A pair of bumps indicates an 'out' arrow and a pair of dents indicates an 'in' arrow. These arrangements are repeated for each face. It happens that every face has a pair of bumps and a pair of dents.

This completes the shape alone matching configuration for the cube. The cube decorated with arrow markings and coloured stripes is isomorphic to its mirror-image. The shape matched cube has a distinct mirror-image. If we adopt again our vantage point on top of a cube looking down we find that the mirror-image cube has a hole from the lower right to the upper left, and so it cannot match with the original type of cube in the same tiling.